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We want to understand now our observables. So we said these are observables, so can we observe them? Can we have a state in which we say, what is the value of L_x , the value of L_y , and the value of L_z . Well, a little caution is necessary because we have states and we have position and momentum operator and they didn't commute and we ended up that we could not tell simultaneously the position and the momentum of a state. So for this angular momentum operators, they don't commute, so a similar situation may be happening.

So I want to explain, for example, or ask, can we have simultaneous eigenstates of L_x , L_y , and L_z ? And the answer is no. And let's see why that happens.

So let's assume we can have simultaneous eigenstates and let's assume, for example, that L_x on that eigenstate ψ is some number $\lambda_x \psi$, and L_y and ψ is equal to $\lambda_y \psi$. Well, the difficulty with this is essentially-- well, we could even say that L_z on ψ is equal to $\lambda_z \psi$. So what is the complication? The complication are those commutators. If you do $L_x L_y$ and ψ , you're supposed to get $i\hbar L_z$ and ψ . And therefore, you're supposed to get $i\hbar \lambda_z \psi$, because it's supposed to be an eigenstate.

But how about the left hand side? The left hand side is $L_x L_y \psi$ minus $L_y L_x \psi$. When L_y acts, it produces a $\lambda_y \psi$, but then L_x acts, it produces a $\lambda_x \lambda_y \psi$ minus $\lambda_y \lambda_x \psi$, which is the same thing, so the left hand side is 0. 0 is equal to $\lambda_z \psi$, so you get λ_z must be 0. If you have a non-trivial state, λ_z should be 0.

By the other commutators-- this can be attained or applied to ψ -- would be 0 again, because each term produces a number and the order doesn't matter. But then it would show that λ_x is 0, and this will show that λ_y is 0. So at the end of the day, if these three things hold, then all of them are 0. λ_x equals λ_y equals λ_z equals 0. So you can have something that is killed by all of the operators, but you cannot have a non-trivial state with non-trivial eigenvalues of these things. So we cannot have-- we cannot tell what is L_x on this state and L_y on this state simultaneously. Any of those two is too much.

So if we can't tell that, what can we tell? So what is the most we can tell about this state Is our

question now. We can tell maybe what is its value of L_x , but then L_y and L_z are undetermined. Or we can tell what is L_z and then L_x and L_y are undetermined, incalculable, impossible in principle to calculate them. So let's see what we can do, and the answer comes from a rather surprising thing, the fact that if you think about what could commute with L_x , L_y , L_z , it should be a rotationally invariant thing, because L_x , L_y , and L_z do rotations. So the only thing that could possibly commute with this thing is something that is rotationally invariant.

The thing that could work out is some thing that is invariant and there are rotations. Now we said, for example, the magnitude of the vector R is invariant under rotation. You rotate the vector, the magnitude is invariant. So we can try the operator L squared, which is proportionate to the magnitude squared, so we define it to be $L_x L_x$ plus $L_y L_y$ plus $L_z L_z$. And we tried, we tried to see if maybe L_x commutes with L squared.

Remember, we had a role for L squared in this differential operator that had the Laplacian, the angular part of the Laplacian was our role for L squared, so L squared is starting to come back. So let's see here-- this is L_x -- now, I'll write the whole thing-- $L_x L_x$ plus $L_y L_y$ plus $L_z L_z$. Now, L_x and L_x commute, so I don't have to bother with this thing, that's 0. But the other ones don't commute. So let's do the distributive law. So this would be an L_x , $L_y L_y$ plus $L_y L_x$, L_y -- this is from the first-- plus L_x , $L_z L_z$ plus $L_z L_x$, L_z . You know, if you don't put these operators in the right order, you don't get the right answer. So I think I did. Yes. It's correct.

Now you use the commutators and hope for the best. So L_x , L_y is $i \hbar L_z$. L_x , L_y is plus $i \hbar L_y L_z$. So far, no signs of canceling, these two things are very different from each other. They don't even appear with a minus sign, so this is not a commutator, but anyway, what is this? L_x with L_z . Well, you should always think cyclically. L_z with L_x is $i \hbar$, so this would be minus $i \hbar L_y L_z$, and this is again L_z with L_x would have been $i \hbar L_y$, so this is minus $i \hbar L_z L_y$, and it better cancel-- yes. This term cancels with the first and this term cancels with this and you get 0.

That's an incredible relief, because now you have a second operator that is measurable simultaneously. You can get eigenstates that are eigenstates of one of the L 's-- for example, L_x and L squared, because they commute, and you won't have the problems you have there. In fact, it's a general theorem of linear algebra that-- we'll see a little bit of that in this course and you'll see it more completely in 805-- that if you have two Hermitian operators that commute, you can find a simultaneous eigenstates of both operators. I mean, eigenstates that are eigenstates of 1, and eigenstates of the second. Simultaneous eigenstates are possible.

So we can find simultaneous eigenstates of these operators, and in fact, you could find simultaneous eigenstates of L_x and L^2 , but given the simplicity of all this, it also means that L_y commutes with L^2 , and that L_z also commutes with L^2 . So you have a choice-- you can choose L_x , L_y , or L_z and L^2 and try to form simultaneous eigenstates from all these operators. Two of them. Let's study those operators as differential operators a little bit.

So x , y , and z are your spherical coordinates and they are $r \sin \theta \cos \phi$, $r \sin \theta \sin \phi$, and $r \cos \theta$. We're trying to calculate the differential operators associated with angular momentum using spherical coordinates. So r is $x^2 + y^2 + z^2$. θ is $\cos^{-1} z/r$ and ϕ is $\tan^{-1} y/x$. And there's something very nice about one angular momentum operator in spherical coordinates, there is only one angular momentum that is very simple-- its rotations about z . Rotations about z don't change the angle θ of spherical coordinates, just change the angle ϕ . r doesn't change. The other rotation, the rotation about x messes up ϕ and θ and all the others are complicated, so maybe we can have some luck and understand what is $d/d\phi$, the $d/d\phi$ operator.

Well, the $d/d\phi$ operator is $d/dy dy/d\phi + d/dx dx/d\phi$ -- the rules of chain rule for partial derivatives-- plus $d/dz dz/d\phi$. But z doesn't depend on ϕ . On the other hand, $dy/d\phi$ is what? $dy/d\phi$, this becomes a $\cos \phi$ -- it's x . $x d/dy$. And $dx/d\phi$ is $-y$. And you say, wow, $x d/dy$ is like $x p_y$ minus $y p_x$, that's a z component of angular momentum!

So indeed, L_z , which is $\hbar/i \times d/dy$ minus $y d/dx$, $\hbar/i$ is because of the p 's-- $x p_y$ minus $y p_x$. And this thing is $d/d\phi$, so L_z , we discover, is just $\hbar/i d/d\phi$. A very nice equation that tells you that the angular momentum in the z direction is associated with its operator.

So I have left us exercises to calculate the other operators that are more messy, and to calculate $L_x L_y$ as well in terms of $d/d\theta$'s and $d/d\phi$'s. And as you remember, angular momentum has units of $\hbar$ and angles have no units, so the units are good and we should find that. So that calculation is left as an exercise, but now you probably could believe that L^2 , which is $L_x L_x + L_y L_y + L_z L_z$ is really $\hbar^2 \times 1/\sin^2 \theta d/d\theta (\sin^2 \theta d/d\theta) + 1/\sin^2 \theta d/d\phi (\sin^2 \theta d/d\phi)$. No, it's-- not $1/\sin^2 \theta$ -- uh, yep. $1/\sin^2 \theta d/d\theta (\sin^2 \theta d/d\theta) + 1/\sin^2 \theta d/d\phi (\sin^2 \theta d/d\phi)$.

So the claim that they had relating the angular momentum operator to the Laplacian is true. But, you know, you now see the beginning of how you calculate these things, but it will be a simple and nice exercise for you to do it.