Integral of a Power Series

We can multiply, add and differentiate power series. Can we integrate them? Yes; as you'd expect, integration of power series is very similar to integration of polynomials. We'll use integration to find a power series expansion for:

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} \quad (x > -1).$$

We know that:

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

So:

$$\begin{aligned}
\ln(1+x) &= \int_0^x (1 - t + t^2 - t^3 + \cdots)dt \\
&= \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\right]_0^x \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots
\end{aligned}$$

Because we began with a power series whose radius of convergence was 1, the radius of convergence of the result will also be 1. This reflects the fact that $\ln(1+x)$ is undefined for $x \le -1$.

Question: If you only use positive values of x is there still a radius of convergence?

Answer: Yes. If $x > 1$ then the numerators x, x^2, x^3, x^4 and so on are increasing exponentially. The denominators 1, 2, 3, 4 ... only grow linearly. So as n goes to infinity, $\frac{x^n}{n}$ will also go to infinity. If the terms of a series go to infinity then the series diverges.

Euler used this kind of power series expansion to calculate natural logarithms much more efficiently than was previously possible.